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CYLINDRICAL MEASURES ON TOPOLOGICAL GROUPS

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Abstract. The concept of cylindrical measures on locally compact Abelian groups is discussed. It is proved that if the convolution of two cylindrical measures μ and ν on G extends to a Radon measure, then there exists an element a belonging to the Bohr compactification of G such that both $\mu * \delta_a$ and $\nu * \delta_a$ have extensions to Radon measures.

1. Cylindrical measures on locally compact Abelian groups. Let G be a locally compact Abelian (LCA) group and let $\Gamma = G^*$ denote its dual group equipped with the standard topology of the uniform convergence on compact subsets of G. We shall use the addive notation for the group operation in G and the multiplicative notation for Γ. Γ_d will denote algebraically the same group Γ, but considered with the discrete topology. Bohr compactification bG of G is defined as the dual of Γ_d. There exists a natural continuous injection G into bG, and in this sense $G \subset bG$. Because $(bG)^* = \Gamma_d$ and $G^* = \Gamma$, the sets of characters of bG and G coincide.

Cylindrical subsets of G. For every n-tuple $\gamma_1, \ldots, \gamma_n$ a function $\gamma = (\gamma_1, \ldots, \gamma_n)$ forms a continuous homomorphism of G into the n-dimensional torus T^n. $\mathscr{B}_\gamma$ will denote the smallest σ-field of subsets of G such that γ is measurable, i.e.

$$\mathscr{B}_\gamma = \{\gamma^{-1}(B)\colon B \in \mathrm{BOREL}(T^n)\}.$$

$\mathscr{B}^0 = \mathscr{B}^0(G)$ stands for the union of all $\mathscr{B}_\gamma$. $\mathscr{B}^0$ is a field of sets but in general it is not a σ-field. By an analogy with the notion of cylindrical subsets of locally convex linear spaces elements of $\mathscr{B}^0$ will be called *cylindrical subsets* of G. Following this analogy, a non-negative normed finitely additive function $\mu\colon \mathscr{B}^0 \to [0, 1]$ such that, for every γ, μ restricted to $\mathscr{B}_\gamma$ is countably additive, is called a *cylindrical measure* (c.m.) on G.

Obviously, the restriction of a probability Radon measure on G to $\mathscr{B}^0$ is a c.m. If a c.m. has any extension to a Radon measure, then it is unique. A c.m. having an extension to a Radon measure will be called a *Radon c.m.*

It should be noted than even if $G = \boldsymbol{R}$ cylindrical sets and c.m.'s in the above sense differ from the objects with the same name studied in the measure theory on locally convex topological vector spaces. This will be easy to see by bijective characteristics of c.m.'s discussed further on (see Proposition 1).

We define now basic operations on c.m'.s.

Image of a c.m. by a homomorphism. If μ_1 is a c.m. on G, and $h\colon G_1 \to G_2$ is a continuous homomorphism of a group G_1 into a group G_2, then $\mu_2(C) = \mu_1(h^{-1}(C))$, $C \in \mathscr{B}^0(G_2)$, defines a c.m. μ_2 on G_2. μ_2 is called an *image of* μ_1 *by a homomorphism* h and denoted by $\mu_2 = h(\mu_1)$.

Product of c.m'.s. Let μ_i be a c.m. on LCA group G_i, $i = 1, 2$. We define the product $\mu = \mu_1 \otimes \mu_2$ as a c.m. on $G = G_1 \times G_2$. First we observe that, for every continuous homomorphism γ of G into a finite-dimensional torus, there exist homomorphisms γ^1 and γ^2 of G_1 and G_2, respectively, into finite-dimensional toruses such that $\mathscr{B}_\gamma \subset \mathscr{B}_{\gamma^1 \otimes \gamma^2}$. Since $\mathscr{B}_{\gamma^1 \otimes \gamma^2} = \mathscr{B}_{\gamma^1} \otimes \mathscr{B}_{\gamma^2}$, we can define a countably additive probability measure on $\mathscr{B}_\gamma$ as the restriction of the product measure $(\mu_1|\mathscr{B}_{\gamma^1}) \otimes (\mu_2|\mathscr{B}_{\gamma^2})$ to $\mathscr{B}_\gamma$ ($(\nu|\mathscr{A})$ denotes the restriction of ν to $\mathscr{A}$). The consistency allows to define a unique c.m. μ on $G = G_1 \otimes G_2$ such that

$$\mu|\mathscr{B}_{\gamma^1 \otimes \gamma^2} = (\mu_1|\mathscr{B}_{\gamma^1}) \otimes (\mu_2|\mathscr{B}_{\gamma^2})$$

for every continuous homomorphism γ^i of G_i $(i = 1, 2)$ into a finite-dimensional torus. We write $\mu = \mu_1 \otimes \mu_2$. It is easy to see that if μ_1 and μ_2 are Radon c.m.'s, then μ is also a Radon c.m.

Convolution of c.m.'s. Let μ_i $(i = 1, 2)$ be a c.m. on a group G. We define the convolution $\mu_1 * \mu_2$ by the formula

$$\mu_1 * \mu_2 = h(\mu_1 \otimes \mu_2),$$

where $h\colon G \times G \to G$ is defined by $h(g_1, g_2) = g_1 + g_2$. We observe that, for every $C \in \mathscr{B}^0$,

$$(\mu_1 * \mu_2)(C) = \int_G \mu_1(C-g)\,\mu_2(dg)$$

and the integral is well-defined since the function $g \to \mu_1(C-g)$ is $\mathscr{B}_\gamma$-measurable and bounded, provided $C \in \mathscr{B}_\gamma$.

We shall discuss now some objects defined by c.m.'s.

Characteristic functional of a c.m. For a c.m. μ we define

$$\hat{\mu}(\gamma) = \int_G \gamma(g)\,\mu(dg), \qquad \gamma \in \Gamma.$$

Note that the integral is well-defined. $\hat{\mu}$ is called the *characteristic functional* of μ. $\hat{\mu}$ is a positive definite normed (p.d.n.) complex value function on Γ which uniquely determines μ. Moreover, we have $(\mu * \nu)\hat{} = \hat{\mu}\hat{\nu}$ for c.m.'s μ and ν.

Radon probability measures on bG. Let $f\colon \Gamma \to C$ be a p.d.n. function. Obviously f is continuous on Γ_d. Thus f is the Fourier transform of a Radon probability measure μ_b on $bG = (\Gamma_d)^*$.

Random character. A stochastic process $\{X(\gamma)\colon \gamma \in \Gamma\}$ with values in the unit circle T such that, for every $\gamma_1, \gamma_2 \in \Gamma$,

$$X(\gamma_1 \gamma_2) = X(\gamma_1) X(\gamma_2) \quad \text{a.s.}$$

is called a *random character*.

Observe that if μ_b is a Radon probability measure on bG, then the formula

$$X(\gamma)(h) = h(\gamma), \qquad h \in bG,$$

defines a random character on a probability space $(bG, \mathrm{BOREL}(bG), \mu_b)$. Every random character $\{X(\gamma)\colon \gamma \in \Gamma\}$ determines a c.m. μ on G as

$$\mu(C) = P\{(X(\gamma_1), \ldots, X(\gamma_n)) \in B\},$$

where $C = \gamma^{-1}(B)$, $\gamma = (\gamma_1, \ldots, \gamma_n)$ and $B \in \mathrm{BOREL}(T^n)$.

Above constructions lead to the following statement:

PROPOSITION 1. *There exist bijections between the following*:

(i) *The set of all c.m.'s on G.*

(ii) *The set of all p.d.n. complex functions defined on Γ (nonnecessary continuous).*

(iii) *The set of all Radon probability measures on bG.*

(iv) *The class of all random characters (under the assumption that processes with the same finite-dimensional distributions are identical).*

To complete this section let us note one more characterization of c.m.'s on LCA groups. Namely, there exists a bijection between c.m.'s on G and positive normed functionals on the space $AP(G)$ of strongly almost periodic functions defined on G. This bijection $\Lambda \leftrightarrow \lambda$ is determined by the equation

$$\Lambda(\gamma) = \hat{\mu}(\gamma),$$

where $\Lambda \in [AP(G)]'$ and satisfies $\Lambda \geqslant 0$ and $\Lambda(1) = 1$.

2. When the convolution of c.m.'s is a Radon measure? In papers [1] and [2] it was proved that if μ and ν are cylindrical measures on a vector space E such that $\mu * \nu$ is a Radon cylindrical measure, then there exists an element $a \in (E')^*$ such that both $\mu * \delta_a$ and $\nu * \delta_{-a}$ are Radon cylindrical measures. Here $(E')^*$ denotes the set of all real linear forms defined on the topological dual E' of E.

Nguyen Van Thu [1] studied the case where E is a Banach space and gave some applications of this fact in the prediction theory of stochastic processes. Rosiński [2] proved this fact in the general case of locally convex topological vector space. In this paper we shall study the case where $E = G$

is an LCA group. As we noted in the last section, c.m.'s on LCA groups form a class of objects different from considered in the measure theory on locally convex topological vector spaces and we cannot simply extend the above mentioned results. There is, however, a formal analogy of that fact and we shall prove the following

THEOREM 1. *Let μ and ν be c.m.'s on an LCA group G. If $\mu * \nu$ is a Radon c.m. on G, then there exists an element $a \in bG$ such that both $\mu * \delta_a$* and *$\nu * \delta_{-a}$ are Radon measures.*

Proof. For a c.m. θ on G, θ_b will stand for the unique extension of θ to a Radon probability measure on bG (see Proposition 1). We set $\lambda = \mu * \nu$. Then we have $\lambda_b = \mu_b * \nu_b$. By the assumption there exists a σ-compact subgroup $K \subset G$, bG, such that $\lambda_b(K) = 1$. Then

$$\int_{bG} \mu_b(K-a)\,\nu_b(da) = 1.$$

Hence there exists an $a \in bG$ such that

$$1 = \mu_b(K-a) = (\mu_b * \delta_a)(K) = (\mu * \delta_a)_b(K),$$

and so $\mu * \delta_a$ extends to a Radon measure on G (see, e.g. [4], Theorem 12, Ch. I). $\nu * \delta_{-a}$ is also a Radon c.m. since

$$\nu_b[(K+a)^c] \leqslant \lambda_b(K^c) + \mu_b[(K-a)^c] = 0.$$

The proof is complete.

By Bochner Theorem and Proposition 1 the following statement is equivalent to just proved Theorem 1:

THEOREM 2. *Let $\varphi, \psi: \Gamma \to C$ be p.d.n. functions defined on an LCA group Γ. If the product $\varphi\psi$ is a continuous function, then there exists a character $\chi \in (\Gamma_d)^*$ such that both functions $\varphi\chi$ and $\psi\chi^{-1}$ are continuous.*

Seemingly weaker but in fact equivalent version of Theorem 2 we obtain taking $\psi = \bar{\varphi}$ in Theorem 2.

THEOREM 3. *Let $\varphi: \Gamma \to C$ be a p.d.n. function defined on an LCA group Γ. If $|\varphi|$ is a continuous function, then there exists a character $\chi \in (\Gamma_d)^*$ such that the function $\varphi\chi$ is continuous.*

Even in the case $\Gamma = \boldsymbol{R}$ the authors do not know any straightforward proof of Theorem 3.

3. Continuity of positive defined functions. This section contains more open questions than completed results. Our general problem we can formulated as follows.

Suppose that $\varphi: G \to C$ is a p.d.n. function defined on an arbitrary topological group G (non-necessary Abelian). Assume that $|\varphi|$ is continuous. What we can say about φ?

We begin with a description of the weakest group topology such that a fixed p.d. function is continuous.

Let $\varphi\colon G \to C$ be a p.d.n. function. Let τ_φ be the weakest topology of G such that all functions of the form

$$G \ni x \to \varphi(y^{-1}xy), \quad y \in G,$$

are continuous (we use the multiplicative notation if G is non-necessarily Abelian). Note that not always τ_φ is a Hausdorff topology.

THEOREM 4. *τ_φ is the weakest group topology on G such that φ is continuous. Moreover, τ_φ is determined by a subbasis of neighbourhoods of the identity of G of the form*

$$U(y, \varepsilon) = \{x \in G\colon |1-\varphi(y^{-1}xy)| < \varepsilon\},$$

where $\varepsilon > 0$ and $y \in G$.

Proof. It is well known that φ can be written in the form

$$\varphi(x) = (U(x)\xi, \xi), \quad x \in G,$$

where $U(\cdot)$ is a unitary representation of G in some Hilbert space H and $\xi \in H$ is a unit cyclic vector of this representation. The strong topology of the group $U(H)$ of the unitary operators on H is a group topology. The proof follows now by the observation that τ_φ is induced by the strong operator topology of $U(H)$ via the representation $U(\cdot)$. The last statement can be easy verified, since

$$\|U(x)U(y)\xi - U(y)\xi\|^2 = 2[1-\operatorname{Re}\varphi(y^{-1}xy)]$$

and the span $\{U(y)\xi\colon y \in G\}$ is dense in H. This ends the proof.

Remark 1. For Abelian groups the topology τ_φ is semimetrizable with a seminorm

$$|x| = [1-\operatorname{Re}\varphi(x)]^{1/2}.$$

Remark 2. For countable groups (non-necessary Abelian) τ_φ is semimetrizable and separable.

We can formulate now a problem concerning p.d.n. functions on arbitrary Abelian groups.

Let G be an Abelian group (now without any topology) and let $\varphi\colon G \mapsto C$ be a p.d.n. function. Does there exist a homomorphism $\chi\colon G \mapsto T$ such that $\varphi\chi$ is continuous in the $\tau_{|\varphi|^2}$ topology?

Note that the positive answer to this question leads to an extension of Theorem 3.

The authors of this note do not know the answer even when $G = Z$. In this case

$$\varphi(n) = \int_T z^n \mu(dz), \quad n \in Z,$$

and the question is: does there exist a complex number $z_0 \in T$ such that the implication "if $|\varphi(n)| \to 1$, then $z_0^n \varphi(n) \to 1$" is true? The answer is "yes" if μ has either nontrivial atomic or absolutely continuous part.

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